

THE THRESHOLD FOR RANDOM (1,2)-QSAT

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The QSAT problem is the quantified version of the SAT problem. We show the existence of a threshold effect for the phase transition associated with the satisfiability of random quantified extended 2-CNF formulas. We consider boolean CNF formulas of the form $\forall X \exists Y \varphi(X, Y)$, where X has m variables, Y has n variables and each clause in φ has one literal from X and two from Y . For such formulas, we show that the threshold phenomenon is controlled by the ratio between the number of clauses and the number n of existential variables. Then we give the exact location of the associated critical ratio c^* . Indeed, we prove that c^* is a decreasing function of α , where α is the limiting value of $m/\log(n)$ when n tends to infinity.

1. Introduction. A significant tool for SAT research has been the study of random instances. It has stimulated fruitful interactions among the areas of artificial intelligence, theoretical computer science, mathematics and statistical physics. Recently there has been a growth of interest in a powerful generalization of the Boolean satisfiability, namely the satisfiability of Quantified Boolean formulas, QBFs. Compared to the well-known propositional formulas, QBFs permit both universal and existential quantifiers over Boolean variables. Thus QBFs allow the modelling of problems having higher complexity than SAT, ranging in the polynomial hierarchy up to PSPACE. These problems include problems from the areas of verification, knowledge representation and logic (see, e.g., [10]).

Models for generating random instances of QBF have been proposed [12, 3]. Problems for which one can combine practical experiments with theoretical studies are natural candidates for first investigations [5]. In this paper, we focus on a certain subclass of closed quantified Boolean formulas,

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which can be seen as quantified extended 2-CNF-formulas. These formulas bear similarities with 2-CNF-formulas, whose random instances have been extensively studied in the literature (see, e.g., [4, 13, 16, 2, 8]). At the same time, the introduction of quantifiers increases the complexity and requires additional parameters for the generation of random instances. More precisely, we are interested in closed formulas in conjunctive normal form (CNF) having two quantifier blocks, namely in formulas of the type $\forall X \exists Y \varphi(X, Y)$, where X and Y denote distinct sets of variables, and $\varphi(X, Y)$ is a conjunction of 3-clauses, each of which containing exactly one universal literal and two existential ones. Evaluating the truth value of such formulas is known to be coNP-complete [11]. In order to generate random instances we have to introduce several parameters. The first one is the pair (m, n) that specifies the number of variables in each quantifier block, i.e., in X and Y . The second one is $L = \lfloor cn \rfloor$, the number of clauses. We shall study the probability that a formula drawn at random uniformly out of this set of formulas evaluates to true as n tends to infinity. We will denote by $\mathbb{P}_{m,c}(n)$ this probability. Thus, we are interested in

$$\lim_{n \rightarrow +\infty} \mathbb{P}_{m,c}(n).$$

Let us recall that the transition from satisfiability to unsatisfiability for random 2-CNF formulas is sharp. Indeed, there is a *critical value* (or a *threshold*) of the ratio of the number of clauses to the number of variables, above which the likelihood of a random 2-CNF-formula being satisfiable vanishes as n tends to infinity, and below which it goes to 1. Moreover, this critical value is known to be 1 (see [4, 13]).

On the one hand observe that, when $m = 1$, a (1,2)-QCNF-formula with L clauses can be seen as the conjunction of two independent 2-CNF-formulas (each of which corresponds to an assignment to the universal variable and has on average $L/2$ clauses). On the other hand, when m is large enough, a random (1,2)-QCNF-formula with $L = \lfloor cn \rfloor$ clauses has essentially strictly distinct universal literals, and then behaves as an existential 2-CNF-formula. Thus, we can easily prove that the transition between satisfiability and unsatisfiability for random (1,2)-QCNF-formulas occurs when c is between 1 and 2. Our main contribution is to identify the scale for m (as a function of n) at which an intermediate and original regime can be observed, $m = \lfloor \alpha \log n \rfloor$. Moreover, at this specific scale in developing further the techniques used by Chvátal and Reed [4], and Goerdts [13], we get the precise location of the threshold as a function of α . Our main result is:

Theorem 1.1 *For any $\alpha > 0$, there exists $c^*(\alpha) > 0$ such that:*

- if $c < c^*(\alpha)$, then $\mathbb{P}_{\lfloor \alpha \ln n \rfloor, c} \xrightarrow{n \rightarrow +\infty} 1$,
- if $c > c^*(\alpha)$, then $\mathbb{P}_{\lfloor \alpha \ln n \rfloor, c} \xrightarrow{n \rightarrow +\infty} 0$.

Moreover, the critical ratio $c^*(\alpha)$ is given by

$$c^*(\alpha) = \begin{cases} 2 & \text{if } \alpha \ln 2 \leq 1 \\ \text{the unique root of } \ln c + \left(\frac{2}{c} - 1\right) \ln(2 - c) = \frac{1}{\alpha} & \text{if } \alpha \ln 2 > 1 \end{cases}$$

Figure 1 shows the evolution of the critical ratio $c^*(\alpha)$ as a function of α .

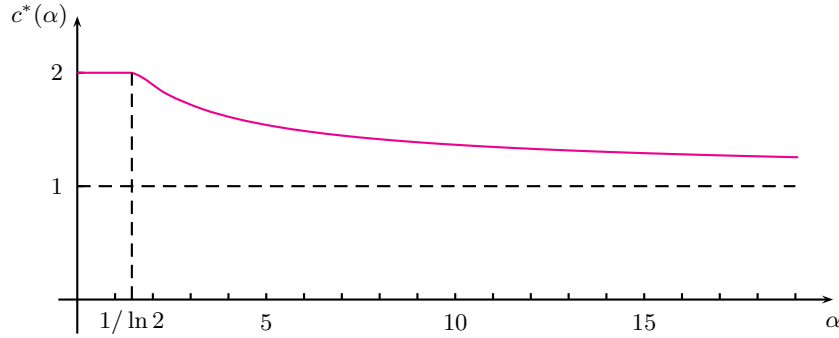


FIG 1. Evolution of the critical ratio values.

The paper is organized as follows. In Section 2 we examine the complexity of deciding the truth value of a (1,2)-QCNF-formula. In order to make the paper self-contained, we give there an alternative proof of the coNP-completeness of this problem. In Section 3 we characterize the truth of (1,2)-QCNF-formulas. We introduce specific substructures, comparable to the ones introduced by Chvátal and Reed in [4]: we define *pure bicycles*, which are necessary to ensure the falsity of a (1,2)-QCNF-formula, and *pure snakes*, whose appearance is sufficient to ensure the falsity. In Section 3.2 we give some enumerative results concerning pure bicycles and snakes, which will be useful for determining the location of the threshold. In Section 4 we present the probabilistic model and we give first estimates for the location of the threshold. In Section 5 we prove our main result, Theorem 1.1. Finally, Section 6 contains the proof of a technical proposition.

2. The complexity of (1,2)-QSAT. A *literal* is a propositional variable or its negation. The *atom* of a literal l is the variable p if l is p or \bar{p} . Literals are said to be *strictly distinct* when their corresponding atoms are

pairwise different. A *clause* is a finite disjunction of literals. A formula is in *conjunctive normal form* (CNF) if it is a conjunction of clauses. A formula is in k -CNF, if any clause consists of exactly k literals. Here we are interested in quantified propositional formulas of the form

$$F = \forall X \exists Y \varphi(X, Y)$$

where $X = \{x_1, \dots, x_m\}$, and $Y = \{y_1, \dots, y_n\}$, and $\varphi(X, Y)$ is a 3-CNF formula, with exactly one universal and two existential literals in each clause. We will call such formulas (1,2)-QCNFs. These formulas can be considered as quantified extended 2-CNF formulas, because deleting the only universal literal in each clause and removing the then superfluous \forall -quantifiers result in an existentially quantified conjunction of binary clauses.

A truth assignment for the existential (resp. universal) variables, X (resp. Y) is a Boolean function $I : X \rightarrow \{0, 1\}$ (resp. $Y \rightarrow \{0, 1\}$), which can be extended to literals by $I(\bar{x}) = 1 - I(x)$

A (1,2)-QCNF formula is *true* (or *satisfiable*) if for every assignment to the variables X , there exists an assignment to the variables Y such that φ is true under this assignment. The exhaustive algorithm which consists in deciding whether for all assignment to the variables X , there exists an assignment to the variables Y such that φ is true provides a first upper bound for the worst case complexity. Indeed, since the satisfiability of a 2-CNF formula can be decided in linear time [1], the evaluation of the formula $\forall X \exists Y \varphi(X, Y)$ can be performed in time $O(2^m \cdot |\varphi|)$, where m is the number of universal variables and $|\varphi|$ denotes the size of φ . Observe that, if m is of the order of $\log n$, then it provides a polynomial time algorithm.

In its full generality the problem (1,2)-QSAT is much harder as stated in the following theorem. This theorem was proved originally in [11]. In order to make the paper self-contained, we give here an alternative proof.

Theorem 2.1 [11] *The evaluation problem (1,2)-QSAT is coNP-complete.*

Proof: To show membership in coNP, guess a vector of truth values v_1, \dots, v_m corresponding to x_1, \dots, x_m . Replace in $\exists Y \varphi(X, Y)$ all free occurrences of any x_i by v_i , remove 0 from the clauses and delete clauses with 1. The resulting formula is a 2-QCNF formula, whose unsatisfiability (i.e. falsity) can be decided in linear time (see [1] for the details).

It remains to be shown that the problem is coNP-hard. We show this by a polynomial-time computable reduction from the satisfiability problem for 3-CNF formulas.

Consider such a formula

$$\alpha: \quad \alpha_1 \wedge \dots \wedge \alpha_n \quad (n \geq 2)$$

over the variables $\{x_2, \dots, x_m\}$ where each α_i is a disjunction of exactly three literals $l_{i,1}$, $l_{i,2}$ and $l_{i,3}$. We construct $\Psi(\alpha)$, a (1,2)-QCNF formula. Then we show that

$$(1) \quad \alpha \text{ is satisfiable} \quad \text{if and only if} \quad \Psi(\alpha) \text{ is false.}$$

The reduction is as follows. We first choose n variables y_1, \dots, y_n , all of which are different from the variables x_2, \dots, x_m occurring in α . We take any minimally unsatisfiable 2-CNF formula with $n+1$ clause, e.g., $\psi = \bigwedge_{i=0}^n \psi_i$ where

$$\psi_i = \begin{cases} \overline{y_1} \vee \overline{y_2} & \text{if } i = 0; \\ y_i \vee \overline{y_{i+1}} & \text{if } i \in \{1, \dots, n-1\}; \\ y_{n-1} \vee y_n & \text{if } i = n. \end{cases}$$

For each clause $\alpha_i = (l_{i,1} \vee l_{i,2} \vee l_{i,3})$ occurring in α , we define

$$\begin{aligned} \psi_{i,1} &= \overline{l_{i,1}} \vee \psi_i, \\ \psi_{i,2} &= \overline{l_{i,2}} \vee \psi_i, \\ \psi_{i,3} &= \overline{l_{i,3}} \vee \psi_i. \end{aligned}$$

Let x_1 be a new variable, i.e., x_1 is different from the ones in $\{y_1, \dots, y_n\}$ and $\{x_2, \dots, x_m\}$. Then

$$\Psi(\alpha): \quad \forall x_1 \forall x_2 \dots \forall x_m \exists y_1 \dots \exists y_n ((x_1 \vee \psi_0) \wedge \bigwedge_{i=1}^n (\psi_{i,1} \wedge \psi_{i,2} \wedge \psi_{i,3})).$$

Obviously, the reduction is polynomial-time computable.

We next prove (1). Observe that the formula resulting from $\Psi(\alpha)$ by any instantiation of the x_i 's is a conjunction of clauses (maybe with repetitions) from ψ . Therefore, since ψ is minimally unsatisfiable, this formula will be unsatisfiable if and only if every clause from ψ occurs.

\Rightarrow : Suppose α is satisfiable. Take an arbitrary truth assignment $I : X \rightarrow \{0, 1\}$, which satisfies α . Then, for all $i = 1, \dots, n$, there is (at least) one $j \in \{1, 2, 3\}$, such that $I(l_{i,j}) = 1$. In the formula $\exists y_1 \dots \exists y_n ((x_1 \vee \psi_0) \wedge \bigwedge_{i=1}^n (\psi_{i,1} \wedge \psi_{i,2} \wedge \psi_{i,3}))$, replace all free occurrences of x_i by $I(x_i)$ for $i = 2, \dots, m$ and x_1 by 0. Observe that, whenever $l_{i,j}$ in $\psi_{i,j}$ (for some $j \in \{1, 2, 3\}$) is true, we get ψ_i after simplification. Therefore, in the existential 2-CNF formula obtained after simplification it remains the clause ψ_0 and at least one copy of each clause ψ_i for every $i = 1, \dots, n$ (the one resulting from $\psi_{i,j}$, for which $I(l_{i,j}) = 1$). Therefore, this formula is unsatisfiable, thus proving that $\Psi(\alpha)$ is false.

\Leftarrow : Suppose $\Psi(\alpha)$ is false. Then, there is a vector of truth values v_1, \dots, v_m corresponding to x_1, \dots, x_m , such that the 2-QCNF formula obtained by replacing all occurrences of any x_i by v_i is unsatisfiable. Since $\psi = \bigwedge_{i=0}^n \psi_i$ is minimally unsatisfiable, and according to the remark above, this means that this resulting formula contains at least one copy of each ψ_i . This copy can only come from a clause $\psi_{i,j}$ for some $j \in \{1, 2, 3\}$. Hence, we can deduce that the assignment $I(x_l) = v_l$ for $l = 1, \dots, m$ sets the literal $l_{i,j}$ to true, and thus satisfies the clause α_i . Hence, this assignment satisfies the formula α . \blacksquare

3. Truth value of (1,2)-QCNF-formulas.

3.1. Pure subformulas. Let us first introduce a notion of purity over sets of universal literals that will be of use to characterize the truth value of (1,2)-QCNF-formulas.

Definition 3.1 *A (multi-)set of literals is pure if it does not contain both a variable x and its negation \bar{x} . By extension, we call a (1,2)-QCNF-formula, $F = \forall X \exists Y \varphi(X, Y)$, pure if the set of universal literals occurring in φ is pure.*

Proposition 3.2 *A (1,2)-QCNF-formula is false if and only if it contains a false pure subformula.*

Proof: One direction is obvious. Suppose that the (1,2)-QCNF-formula $F = \forall X \exists Y \varphi(X, Y)$ is false. Then, there is an assignment I to the universal variables X such that for all assignment to Y , φ evaluates to false. Consider the subformula of F obtained in keeping only the clauses for which the universal literal is assigned 0 by I , and deleting the other ones. This subformula is pure (it cannot contain both a clause with a universal variable x and another with \bar{x} since either x or \bar{x} is assigned 1 by I), and is false by the choice of I . \blacksquare

Now observe that the truth value of a pure (1,2)-QCNF-formula F is the same as the truth value of the existential 2-CNF formula F_Y obtained in removing the universal literal in each clause and then deleting the universal quantifiers. Therefore, we can appeal to the work of Chvátal and Reed [4] in order to identify substructures that are sufficient (respectively, necessary) to ensure falsity. On the one hand Chvátal and Reed exhibited elementary unsatisfiable 2-CNF-formulas, called *snakes*. On the other hand they identified extremal substructures, called *bicycles*, that appear in any unsatisfiable 2-CNF-formula. Thus, we can define *pure snakes* and *pure bicycles*.

Definition 3.3 A pure snake of length $s + 1 \geq 4$, with $s + 1 = 2t$, is a set of $s + 1$ clauses C_0, \dots, C_s which have the following structure: there is a sequence of s strictly distinct existential literals w_1, \dots, w_s , and a pure sequence of $s + 1$ universal literals v_0, \dots, v_s such that, for every $0 \leq r \leq s$, $C_r = (v_r \vee \overline{w_r} \vee w_{r+1})$ with $w_0 = w_{s+1} = \overline{w_t}$.

Definition 3.4 A pure bicycle of length $s + 1 \geq 3$, is a set of $s + 1$ clauses C_0, \dots, C_s which have the following structure: there is a sequence of s strictly distinct existential literals w_1, \dots, w_s , and a pure sequence of $s + 1$ universal literals v_0, \dots, v_s such that, for $0 < r < s$, $C_r = (v_r \vee \overline{w_r} \vee w_{r+1})$, $C_0 = (v_0 \vee u \vee w_1)$ and $C_s = (v_s \vee \overline{w_s} \vee v)$ with literals u and v chosen from $w_1, \dots, w_s, \overline{w_1}, \dots, \overline{w_s}$ with $(u, v) \neq (\overline{w_s}, w_1)$.

Thus, we get the following proposition.

Proposition 3.5

- Every (1,2)-QCNF-formula that contains a pure snake is false.
- Every (1,2)-QCNF-formula that is false, contains a pure bicycle.

3.2. Enumerative results.

Proposition 3.6 Let m be the number of universal variables and let n be the number of existential variables we can choose from.

- The number of snakes of length $s + 1$ is

$$(2) \quad (n)_s 2^s d(m, s + 1)$$

where

$$(3) \quad d(m, s + 1) = \sum_{k=1}^{\min(m, s+1)} \binom{m}{k} \cdot 2^k \cdot \mathcal{S}(s + 1, k) \cdot k!$$

with $\mathcal{S}(m, k)$ denoting the Stirling number of the second kind, and $(n)_s = (n - 1) \cdots (n - s + 1)$.

- Given a pure snake A_0 of length $s + 1 = 2t$. For every $1 \leq i \leq 2t - 1$, let $N_{m,s}(i)$ denote the number of pure snakes B of length $s + 1$ such that A_0 and B share exactly i clauses. Then for $1 \leq i \leq t - 1$

$$(4) \quad N_{m,s}(i) \leq 2(s + 1)^3 \left[\sum_{h=1}^{2t} \left(\frac{(s + 1)^3}{n - s} \right)^h \right] (n)_{s-i} 2^{s-i} d(m, s + 1 - i)$$

and for $t \leq i \leq 2t - 1$

$$(5) \quad N_{m,s}(i) \leq 2(s+1)^3 \left[\sum_{h=0}^{2t} \left(\frac{(s+1)^3}{n-s} \right)^h \right] (n)_{s-i} 2^{s-i} d(m, s+1-i)$$

hold.

- The number of bicycles of length $s+1$ is

$$(6) \quad [(2s)^2 - 1](n)_s 2^s d(m, s+1)$$

Proof: Given a literal w , let $|w|$ denote its underlying variable. Observe that a snake of length $s+1 = 2t$ contains s distinct variables. Moreover, every variable $|w_i|$ appearing in a snake occurs exactly twice (once positively and once negatively), except for $|w_0|$ which occurs four times (twice positively and twice negatively). This special variable will be called the *double point* of the snake. A snake can be described by a (circular) sequence of existential literals $w_0, w_1, \dots, w_s(w_0)$ (with $w_0 = \overline{w_t}$), together with the corresponding pure sequence of universal literals v_0, v_1, \dots, v_s .

Choosing a snake of length $s+1$ comes down to choose a sequence of s strictly distinct literals w_1, \dots, w_s , and then choose the pure sequence of $s+1$ universal literals v_0, \dots, v_s (they are not necessarily distinct but no literal can be the complement of another). Let $d(m, s+1)$ be the number of pure sequences of literals of length $s+1$, having a set of m variables from which the literals can be built. Let us recall that $\mathcal{S}(m, k) \cdot k!$ is the number of applications from a set of m elements onto a set of k elements. A pure sequence of literals of length $s+1$ is obtained by exactly one sequence of choices of the following choosing process.

1. Choose the number k of different variables occurring in the sequence.
2. Choose the k variables.
3. For each such variable, choose whether it occurs positively or negatively.
4. Choose their places in the sequence.

This gives the announced number of snakes.

Given a pure snake A_0 of length $s+1 = 2t$. Let $N_{m,s}(i)$ be the number of pure snakes B of length $s+1$ such that A_0 and B share exactly i clauses. If $i \leq 2t - 1$, this number can be decomposed as

$$N_{m,s}(i) = \sum_{j \geq i+1} N_{m,s}(i, j)$$

where $N_{m,s}(i, j)$ is the number of pure snakes B such that A_0 and B share exactly i clauses and j variables. In the rest of the proof, for more readability we omit the subscripts m, s in $N_{m,s}(i, j)$, thus writing $N(i, j)$. Now we are looking for upper bounds on the $N(i, j)$.

Let us note that the intersection of A_0 and B can be read on the (circular) sequence of literals $w_0, w_1, \dots, w_t, \dots, w_s(w_0)$, where $w_t = \overline{w_0}$. In order to get i clauses and j variables in common, one has to choose $k = (j - i)$ blocks of consecutive literals in this sequence. We make a case distinction according to whether the two snakes A_0 and B have the same double point or not.

- $N^a(i, j)$ denotes the number of pure snakes B of length $s + 1$ such that A_0 and B share exactly i clauses and j variables, and have the same double point $|w_0|$,
- $N^b(i, j)$ denotes the number of pure snakes B of length $s + 1$ such that A_0 and B share exactly i clauses and j variables, and do not have the same double point.

Thus $N(i, j) = N^a(i, j) + N^b(i, j)$.

Let us first consider $N^a(i, j)$. Observe that in the special case when $j = i + 1$ (only one block), and A_0 and B have the same double point, then i is necessarily equal to or larger than t . Therefore,

$$(7) \quad \text{for } 1 \leq i \leq t - 1, \quad N^a(i, i + 1) = 0 .$$

In the general case, to count $N^a(i, j)$, we perform the following sequence of choices :

- (i) the intersection $A_0 \cap B$ such that it has i clauses and j variables,
- (ii) the sequence of strictly distinct existential literals that are in $B \setminus (A_0 \cap B)$
- (iii) the places of the k blocks of $A_0 \cap B$ among the literals chosen in (ii),
- (iv) the universal literals occurring in the clauses of $B \setminus (A_0 \cap B)$.

Step (i). To build the intersection $A_0 \cap B$, we choose $2k$ literals in the sequence representing A_0 . They represent the first and last literals of the k blocks of $A_0 \cap B$. The first literal is chosen after or at w_0 . To define completely the intersection, we need to know whether this first literal is the beginning or the end of a block, so we get at most $2 \binom{s+1}{2k} \leq (s+1)^{2k}$ possible choices.

Step (ii). Notice that $|w_0|$ is the double point of B . So, it remains only to choose a sequence of $s - (j - 1)$ strictly distinct literals. Thus, we have at most $(n)_{s+1-j} 2^{s+1-j}$ possible choices.

Step (iii). We need to choose how the k blocks will be plugged among the “remaining literals” chosen in (ii). This leads to at most $(s+1)^k$ possible choices.

Step (iv). There are $s + 1 - i$ universal literals to choose, and they must be chosen in a pure way. So, there are at most $d(m, s + 1 - i)$ choices.

Thus, since $k = j - i$ we obtain that for $1 \leq i \leq 2t - 1$, $j \geq i + 1$

$$(8) \quad N^a(i, j) \leq (n - s) \left(\frac{(s + 1)^3}{n - s} \right)^{j-i} (n)_{s-i} 2^{s-i} d(m, s + 1 - i).$$

The enumeration of $N^b(i, j)$ differs from the one of $N^a(i, j)$ only at step (ii). Indeed, when B does not have $|w_0|$ as a double point, at step (ii) we have first to choose a sequence of $s - j$ strictly distinct literals (thus having determined the s variables occurring in B), and then choose one of these s variables as the double point. Hence, we have at most $s(n)_{s-j} 2^{s-j}$ choices. Thus, we get for $1 \leq i \leq 2t - 1$ and $j \geq i + 1$

$$(9) \quad N^b(i, j) \leq s \left(\frac{(s + 1)^3}{n - s} \right)^{j-i} (n)_{s-i} 2^{s-i} d(m, s + 1 - i).$$

Then, equation (4) follows from (7), (8) and (9) while (5) follows from (8) and (9).

The enumeration of bicycles is similar to the one of snakes. We just have to choose in addition u and v among $w_1, \dots, w_s, \overline{w_1}, \dots, \overline{w_s}$ such that $(u, v) \neq (\overline{w_s}, w_1)$. This explains the extra factor, $[(2s)^2 - 1]$, in (6)

■

4. Location of the transition for (1,2)-QSAT. We consider formulas built on n universal variables and m existential variables. Thus we have $N = m \binom{n}{2} 2^3 = 4mn(n - 1)$ different clauses at hand. We may establish our result in considering random formulas obtained by taking each one of the N possible clauses independently from the others with probability $p \in]0, 1[$. Let $c > 0$, it is well known, see for instance [14, Sections 1.4 and 1.5], that the threshold obtained in this model translates to the model alluded to in the introduction – in which $L = \lfloor cn \rfloor$, distinct clauses are picked uniformly at random among all the N possible choices –, when $p = \frac{L}{4mn(n-1)}$. Thus, from now on we shall always suppose that $p = \frac{c}{4mn}$, and we continue to denote by $\mathbb{P}_{m,c}(n)$ the probability that a random formula in this model is satisfiable. We are interested in studying $\lim_{n \rightarrow +\infty} \mathbb{P}_{m,c}(n)$ as a function of the parameters m and c . Any value of c such that $\mathbb{P}_{m,c}(n) \rightarrow 1$ (resp. such that $\mathbb{P}_{m,c}(n) \rightarrow 0$) gives a lower (resp. upper) bound for the threshold effect associated to the phase transition.

Let us recall that the 2-SAT property exhibits a sharp transition, with a critical value equal to 1 (see [4] and [13]). From this result it is easy to deduce that the phase transition from satisfiability to unsatisfiability for (1,2)-QCNF formulas occurs when $1 \leq c \leq 2$.

Proposition 4.1 *Let $m = m(n)$ be any sequence of integers.*

- *If $c < 1$ then $\mathbb{P}_{m,c}(n) \xrightarrow{n \rightarrow \infty} 1$.*
- *If $c > 2$ then $\mathbb{P}_{m,c}(n) \xrightarrow{n \rightarrow \infty} 0$.*

Proof: Let F be a random (1,2)-QCNF-formula. Let us consider F_t , the 2-CNF formula obtained from F by setting all the variables x_1, \dots, x_m to *true* and omitting all quantifiers. If F is satisfiable, then so is F_t . Notice that F_t can be obtained by picking independently each possible 2-clause with probability

$$q(n) = 1 - (1 - p(n))^m = \frac{c}{4n} + O\left(\frac{1}{n^2}\right).$$

Thus the average number of clauses in F_t is equal to

$$4 \binom{n}{2} \cdot q \sim c/2 \cdot n.$$

It follows from the threshold of 2-SAT [4, 13] that F_t is unsatisfiable with probability tending to 1 if $c > 2$. Thus, the same holds for F .

Now, we look at the existential part of the formula, F_Y . Observe that if F_Y is satisfiable, then so is F . In F_Y , each of the $4 \binom{n}{2}$ 2-clauses appears independently with probability

$$q'(n) = 1 - (1 - p(n))^{2m} = \frac{c}{2n} + O\left(\frac{1}{n^2}\right).$$

Therefore, the threshold of 2-SAT tells us that when $c < 1$, the formula F_Y is satisfiable with probability tending to one. The same holds for F . ■

5. Proof of the main result.

5.1. *General inequalities.* Let B_s and X_s be respectively the number of pure bicycles and pure snakes of length $s + 1$ in a random (1,2)-QCNF formula. Let us recall that in such a formula, each clause is chosen with probability $p = \frac{c}{4mn}$. Hence, if $\mathbb{E}_{m,c}(B_s)$ and $\mathbb{E}_{m,c}(X_s)$ denote the average

number of bicycles and snakes of length $s + 1$ in a random (1,2)-QCNF formula, we get from (2), (3) and (6) the following two equations:

$$(10) \quad \mathbb{E}_{m,c}(X_s) = p^{s+1}(n)_s 2^s d(m, s+1)$$

$$(11) \quad \mathbb{E}_{m,c}(B_s) = \mathbb{E}_{m,c}(X_s)((2s)^2 - 1).$$

In order to prove that c^* is the critical value for the (decreasing) satisfiability property for (1,2)-QCNF-formulas, we will use two sequences of inequalities. The first one follows from Proposition 3.5 and Markov inequality applied on the number of bicycles. We have

$$(12) \quad 1 - \mathbb{P}_{m,c}(n) \leq \Pr\left(\sum_{s \geq 2} B_s \geq 1\right) \leq \sum_{s \geq 2} \mathbb{E}_{m,c}(B_s).$$

The second one is obtained in considering the number of snakes. Proposition 3.5 and a general exponential inequality given in [14, Theorem 2.18 ii)] show that for any $s \geq 3$

$$(13) \quad \mathbb{P}_{m,c}(n) \leq \Pr(X_s = 0) \leq \exp\left(-\frac{\mathbb{E}_{m,c}(X_s)}{1 + \sum_{i=1}^s N_{m,s}(i)p^{s+1-i}}\right)$$

Finally, recall that we can suppose that $1 < c < 2$, according to Proposition 4.1.

5.2. *When the critical ratio is equal to 2.* Let us start with a proposition which enables to control the mean number of bicycles for any c in $]1, 2[$.

Proposition 5.1 *For any $1 < c < 2$, the following statements hold when n tends to infinity*

- if $m \leq \frac{\ln n}{\ln 2}$ then $\sum_{s \geq 2} \mathbb{E}_{m,c}(B_s) = o(1)$
- if $m = \lfloor \alpha \ln n \rfloor$ with $\alpha \ln 2 > 1$ then $\sum_{s \geq 2^{\frac{\alpha \ln 2 - 1}{\ln 2 - \ln c}} \ln n} \mathbb{E}_{m,c}(B_s) = o(1).$

Proof: Let us recall that the coefficient $d(m, s+1)$ occurring in $\mathbb{E}_{m,c}(B_s)$ is the number of pure sequences of literals of length $s+1$, when we have m variables from which the literals can be built. Note that $d(m, s+1)$ is bounded from above by $2^{\min\{m, s+1\}}$ times the number of applications from $\{1, \dots, s+1\}$ to $\{1, \dots, m\}$. Therefore,

$$(14) \quad d(m, s+1) \leq 2^{\min\{m, s+1\}} m^{s+1}.$$

From (11), it follows that if $s < m$ then $\mathbb{E}_{m,c}(B_s) \leq \frac{c^{s+1}s^2}{n}$. Thus

$$(15) \quad \sum_{s < m} \mathbb{E}_{m,c}(B_s) \leq \left(\frac{c}{c-1}\right) m^2 \frac{c^m}{n}.$$

If $s \geq m$, then (14) gives $\mathbb{E}_{m,c}(B_s) \leq \left(\frac{c}{2}\right)^{s+1} s^2 \frac{2^m}{n}$. When $0 < x < 1$ and $r \geq 2$, standard computations show that

$$(16) \quad \sum_{s=r}^{\infty} s^2 x^s \leq r^2 \frac{x^r}{(1-x)^3}.$$

Hence we get

$$(17) \quad \sum_{s \geq r} \mathbb{E}_{m,c}(B_s) \leq \frac{c 2^m r^2 \left(\frac{c}{2}\right)^r}{n(1-c/2)^3}.$$

The proof of Proposition 5.1 is now an easy consequence of (15) and (17). \blacksquare

Theorem 1.1 when $\alpha \ln 2 \leq 1$ follows from Proposition 5.1, inequality (12) and Proposition 4.1.

In the sequel, we consider the case where $m = \lfloor \alpha \ln n \rfloor$, with $\alpha > 1/\ln 2$.

5.3. The critical ratio as a function of α . The main difficulty when dealing with $\mathbb{E}_{m,c}(B_s)$ and $\mathbb{E}_{m,c}(X_s)$ is to handle the coefficient $d(m, s+1)$ given in Proposition 3.6

$$d(m, s+1) = \sum_{k=1}^{\min(m, s+1)} \binom{m}{k} \cdot 2^k \cdot \mathcal{S}(s+1, k) \cdot k!.$$

First, let us denote for $1 \leq k \leq \min(m, s+1)$

$$(18) \quad G_{m,c}(k, s+1) = 2^s (n)_s \binom{m}{k} 2^k \mathcal{S}(s+1, k) k! \left(\frac{c}{4mn}\right)^{s+1}.$$

From (10) and (11), the behavior of $\mathbb{E}_{m,c}(X_s)$ and $\mathbb{E}_{m,c}(B_s)$ is clearly governed by the coefficients $G_{m,c}(k, s+1)$. Indeed, since $p = \frac{c}{4mn}$ we get

$$(19) \quad \mathbb{E}_{m,c}(B_s) = \sum_{k=1}^{\min(m, s+1)} G_{m,c}(k, s+1) ((2s)^2 - 1) = ((2s)^2 - 1) \mathbb{E}_{m,c}(X_s)$$

Second, we will need better bounds than the one given in (14). We will use well-known estimates for binomial coefficients. If $1 \leq b \leq a$, then the following inequalities hold:

$$(20) \quad \sqrt{\frac{1}{a}} \left(\frac{a}{b}\right)^b \cdot \left(\frac{a}{a-b}\right)^{a-b} \leq \binom{a}{b} \leq \left(\frac{a}{b}\right)^b \cdot \left(\frac{a}{a-b}\right)^{a-b}.$$

Then, from [15], we have the following bounds for Stirling numbers of the second kind. There exist $K > 0$ and $K' > 0$ such that, for $1 \leq b \leq a$, the following inequalities hold:

$$(21) \quad K \sqrt{\frac{b}{a}} \left(\frac{e^{x_0} - 1}{x_0}\right)^b \left(\frac{a}{e}\right)^a x_0^{b-a} \leq b! S(a, b) \leq K' \sqrt{b} \left(\frac{e^{x_0} - 1}{x_0}\right)^b \left(\frac{a}{e}\right)^a x_0^{b-a}$$

where $x_0 > 0$ is a function of b/a defined implicitly for $b < a$ by $1 - e^{-x_0} = \frac{b}{a} x_0$, and for $a = b$ by $x_0 = 0$. The conventions are that $0^0 = 1$ and $\frac{e^0 - 1}{0} = 1$.

By using these precise results, already used in [9] and [5], it appears that the behaviour of the coefficients $G_{m,c}(k, s+1)$ and so the one of the average number of snakes or bicycles, is governed by a continuous function of several real variables. From (18), (20) and (21) we obtain:

Proposition 5.2 *There exist $A > 0$ and $B > 0$ such that for any $c > 0$, for every positive integers n, m, s and k such that $k \leq \min(m, s+1)$:*

$$(22) \quad \frac{A(n)_s \sqrt{k}}{n^s \sqrt{m(s+1)}} n^{g_{\frac{m}{\ln n}, c}(\frac{k}{\ln n}, \frac{s+1}{\ln n})} \leq G_{m,c}(k, s+1) \leq B \sqrt{m} n^{g_{\frac{m}{\ln n}, c}(\frac{k}{\ln n}, \frac{s+1}{\ln n})}$$

where $g_{\alpha,c}$ is the continuous function on $\mathcal{D}_\alpha = \{(\beta, \gamma) \mid 0 < \beta \leq \alpha \text{ and } \beta \leq \gamma\}$ defined for $0 < \beta < \gamma$ by

$$(23) \quad g_{\alpha,c}(\beta, \gamma) = \ln \left[\frac{1}{e} \left(\frac{c\gamma}{2ex_0\alpha} \right)^\gamma \cdot \frac{\alpha^\alpha}{\beta^\beta (\alpha - \beta)^{\alpha - \beta}} \cdot 2^\beta \cdot (e^{x_0} - 1)^\beta \right],$$

$$\text{with } 1 - e^{-x_0} = \frac{\beta}{\gamma} x_0 \text{ and } g_{\alpha,c}(\beta, \beta) = \ln \left[\frac{1}{e} \left(\frac{c}{e\alpha} \right)^\beta \cdot \frac{\alpha^\alpha}{(\alpha - \beta)^{\alpha - \beta}} \right].$$

Recall that we have taken $m = \lfloor \alpha \ln n \rfloor$. Observe that the second part of Proposition 5.1 together with (11) indicates that long snakes, and similarly long bicycles, of length $\gg \ln n$, have asymptotically no chance to appear when $\alpha > 1/\ln 2$ and $c \in]1, 2[$. Therefore, in our study we will focus on snakes of length proportional to $\ln n$. Hence, let us set $\beta = k/\ln n$, $\gamma = (s+1)/\ln n$. The following result will point out for each α , the values of k and s that contribute the most to the average number. Indeed we will prove the following central result :

Proposition 5.3 *Let $1 < c < 2$, and for any α let \mathcal{D}_α be the following domain*

$$\mathcal{D}_\alpha = \{(\beta, \gamma) \mid 0 < \beta \leq \alpha \text{ and } \beta \leq \gamma\}.$$

The function $g_{\alpha,c}$ defined by (23) has a global maximum on \mathcal{D}_α , given by its unique stationarity point in \mathcal{D}_α . More precisely

$$(24) \quad \max_{\mathcal{D}_\alpha} g_{\alpha,c}(\beta, \gamma) = g_{\alpha,c}(\hat{\beta}(\alpha, c), \hat{\gamma}(\alpha, c)) = \alpha H(c) - 1$$

$$\text{with } \hat{\beta} = \frac{2\alpha(c-1)}{c}, \hat{\gamma} = \frac{-2\alpha \ln(2-c)}{c}, \quad H(c) = \ln c + \left(\frac{2}{c} - 1\right) \ln(2-c).$$

Moreover, for any domain $V_\alpha \subset \mathcal{D}_\alpha$ such that $(\hat{\beta}, \hat{\gamma}) \notin \overline{V_\alpha}$ then

$$(25) \quad \max_{V_\alpha} g_{\alpha,c}(\beta, \gamma) < \alpha H(c) - 1.$$

The proof of this result is rather technical, so we postpone it to the next section.

Now we can prove Theorem 1.1 when $\alpha \ln 2 > 1$. In other words that, when $\alpha \ln 2 > 1$, the critical ratio $c^*(\alpha)$ is the unique root of $\alpha H(c) = 1$. For this, we will use two corollaries of Proposition 5.2 and Proposition 5.3.

Corollary 5.4 *Let $\alpha > 1/\ln 2$ and $c < 2$ be such that $\alpha H(c) < 1$. Then, as n tends to infinity*

$$\sum_{s \geq 2} \mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(B_s) = o(1).$$

Proof: From Proposition 5.1, we have $\sum_{s \geq 2^{\frac{\alpha \ln 2 - 1}{\ln 2 - \ln c} \ln n}} \mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(B_s) = o(1)$.

Then, from (19), the upper bound (22) and (24) we get

$$\sum_{s < 2^{\frac{\alpha \ln 2 - 1}{\ln 2 - \ln c} \ln n}} \mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(B_s) \leq B' \alpha^{3/2} (\ln n)^{9/2} \left[\frac{\alpha \ln 2 - 1}{\ln 2 - \ln c} \right]^3 n^{\theta_n},$$

with $\theta_n \rightarrow g_{\alpha,c}(\hat{\beta}, \hat{\gamma}) = \alpha H(c) - 1 < 0$. Therefore, $\sum_{s < 2^{\frac{\alpha \ln 2 - 1}{\ln 2 - \ln c} \ln n}} \mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(B_s) = o(1)$. ■

With (12), this corollary proves that, when $\alpha \ln 2 > 1$ and for any $c < c^*(\alpha)$, we have $\mathbb{P}_{m,c}(n) = 1 - o(1)$.

In considering (13) with $s + 1 = \lfloor \hat{\gamma} \ln n \rfloor = 2t$, it will follow easily from the corollary given below that, when $\alpha \ln 2 > 1$, for any $c > c^*(\alpha)$ we have

$\mathbb{P}_{m,c}(n) = o(1)$. This will end the proof of Theorem 1.1. Note that the coefficients $N_{m,s}(i)$ appearing in the following corollary are the ones defined in Proposition 3.6.

Corollary 5.5 *Let $\alpha > 1/\ln 2$ and $c < 2$ be such that $\alpha H(c) > 1$, and let $s+1 = \lfloor \hat{\gamma} \ln n \rfloor$. Then there exist $0 < \delta < 2(\alpha H(c) - 1)$, $C > 0$ and $D > 0$ such that*

$$\mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(X_s) \geq C n^{\alpha H(c) - 1 - \frac{\delta}{2}}$$

and

$$\sum_{i=1}^s N_{\lfloor \alpha \ln n \rfloor, s}(i) \left(\frac{c}{4mn} \right)^{s+1-i} \leq D n^{\alpha H(c) - 1 - \frac{2\delta}{3}}.$$

Proof: From (25) in Proposition 5.3, we first choose $\delta \in]0, 2(\alpha H(c) - 1)[$ such that

$$\max_{\{(\beta, \gamma) \text{ s.t. } \gamma < \frac{\hat{\gamma}}{2}\} \cap \mathcal{D}_\alpha} g_{\alpha, c}(\beta, \gamma) \leq \max_{\mathcal{D}_\alpha} g_{\alpha, c}(\beta, \gamma) - \delta.$$

Again in using (19) and the lower bound in (22), we can find $C > 0$ such that for $s+1 = \lfloor \hat{\gamma} \ln n \rfloor$

$$\mathbb{E}_{\lfloor \alpha \ln n \rfloor, c}(X_s) \geq C n^{g_{\alpha, c}(\hat{\beta}, \hat{\gamma}) - \frac{\delta}{2}}.$$

As $g_{\alpha, c}(\hat{\beta}, \hat{\gamma}) = \alpha H(c) - 1$, the first assertion is proved.

Then, with $p = \frac{c}{4mn}$, from (4) and (5) we get first for $1 \leq i < t$

$$N_{m,s}(i) p^{s+1-i} \leq 2(s+1)^3 \sum_{k=1}^{\min(m, s+1)} \left[\sum_{h=1}^{2t} \left(\frac{(s+1)^3}{n} \right)^h \right] G_{m,c}(k, s+1-i)$$

and second for $t \leq i \leq 2t-1$

$$N_{m,s}(i) p^{s+1-i} \leq 2(s+1)^3 \sum_{k=1}^{\min(m, s+1)} \left[\sum_{h=0}^{2t} \left(\frac{(s+1)^3}{n} \right)^h \right] G_{m,c}(k, s+1-i).$$

At last, in using (22) with $s+1 = \lfloor \hat{\gamma} \ln n \rfloor$ and with our choice for δ we obtain

$$\begin{aligned} \sum_{i=1}^{t-1} N_{\lfloor \alpha \ln n \rfloor, s}(i) \left(\frac{c}{4mn} \right)^{s+1-i} &\leq D_1 (\ln n)^{\frac{15}{2}} n^{\alpha H(c) - 2} \\ \sum_{i=t}^{2t-1} N_{\lfloor \alpha \ln n \rfloor, s}(i) \left(\frac{c}{4mn} \right)^{s+1-i} &\leq D_2 (\ln n)^{\frac{9}{2}} n^{\alpha H(c) - 1 - \delta}. \end{aligned}$$

■

6. Proof of Proposition 5.3. Let us recall that for any $1 < c < 2$ and $\alpha > 0$, we consider the domain $\mathcal{D}_\alpha = \{(\beta, \gamma) \mid 0 < \beta \leq \alpha \text{ and } \beta \leq \gamma\}$ for the function $g_{\alpha,c}$ given from (23) by

$$(26) \quad g_{\alpha,c}(\beta, \gamma) = -1 + \alpha \ln \alpha - (\alpha - \beta) \ln(\alpha - \beta) + \gamma \ln \left[\frac{c\gamma}{2ex_0\alpha} \right] + \beta \ln \left[\frac{2(e^{x_0} - 1)}{\beta} \right]$$

$$(27) \quad g_{\alpha,c}(\beta, \beta) = -1 + \alpha \ln \alpha - (\alpha - \beta) \ln(\alpha - \beta) + \beta \ln \left[\frac{c}{e\alpha} \right]$$

with x_0 defined implicitly when $0 < \beta < \gamma$ by

$$(28) \quad 1 - e^{-x_0} = \frac{\beta}{\gamma} x_0$$

In the sequel, we shall write g for $g_{\alpha,c}$ and \mathcal{D} for \mathcal{D}_α .

Proposition 5.3 tells us that g has a strict and global maximum on \mathcal{D} which is equal to $\alpha H(c) - 1$ with $H(c) = \ln c + \left(\frac{2}{c} - 1\right) \ln(2 - c)$. The proof of Proposition 5.3 follows from the following claim :

Claim 6.1 *For any $1 < c < 2$ and $\alpha > 0$,*

1. *for every fixed β with $0 < \beta \leq \alpha$, the function $\gamma \mapsto g(\beta, \gamma)$ is strictly concave on $[\beta, +\infty[$ with a strict maximum at $\gamma_\beta = \frac{2\alpha}{c} \ln \left(\frac{2\alpha}{2\alpha - \beta c} \right)$.*
2. *the function $\beta \mapsto g(\beta, \gamma_\beta)$ is strictly concave on $]0, \alpha]$ with a maximum at $\hat{\beta} = \frac{2\alpha(c-1)}{c}$, then with $\hat{\gamma} := \gamma_{\hat{\beta}} = \frac{-2\alpha \ln(2-c)}{c}$, $g(\hat{\beta}, \hat{\gamma}) = \alpha H(c) - 1$.*

Proof: For the first point of this claim we compute, from (26) and (28), the partial derivatives of g with respect to γ . We get

$$(29) \quad \frac{\partial g}{\partial \gamma}(\beta, \gamma) = \ln \left(\frac{c\gamma}{2x_0\alpha} \right) \quad \text{and} \quad \frac{\partial^2 g}{\partial \gamma^2}(\beta, \gamma) = \frac{\gamma - \beta x_0}{\gamma(\gamma - \beta(x_0 + 1))}.$$

With (28) we first observe that

$$(30) \quad \gamma - \beta x_0 = \gamma e^{-x_0} > 0.$$

Then

$$\begin{aligned} \gamma - \beta(x_0 + 1) &= \gamma - \beta x_0 - \beta \\ &= \gamma e^{-x_0} - \beta \\ &= \gamma e^{-x_0} - \frac{\gamma(1 - e^{-x_0})}{x_0} \\ &= \frac{\gamma}{x_0} (x_0 e^{-x_0} - 1 + e^{-x_0}) \end{aligned}$$

let $\varphi(x) = xe^{-x} - 1 + e^{-x}$. The function φ is decreasing with $\varphi(0) = 0$. Hence, $\varphi(x_0) < 0$ and

$$(31) \quad \gamma - \beta(x_0 + 1) < 0.$$

From the second identity in (29), (30) and (31) we conclude that $\frac{\partial^2 g}{\partial \gamma^2}(\beta, \gamma) < 0$. The strict concavity of $\mapsto g(\beta, \gamma)$ follows. Then the first identity in (29) and (28) give the expected formula for the unique extremum, indeed we obtain

$$(32) \quad \gamma_\beta = \frac{2x_0\alpha}{c} = \frac{2\alpha}{c} \ln\left(\frac{2\alpha}{2\alpha - \beta c}\right) \text{ and } e^{x_0} - 1 = \frac{\beta c}{2\alpha - \beta c}.$$

For the second point of the claim, observe that with (26) we have :

$$g(\beta, \gamma) = -1 + \gamma \ln\left[\frac{c\gamma}{2x_0\alpha}\right] - \gamma + \alpha \ln \alpha - (\alpha - \beta) \ln(\alpha - \beta) + \beta \ln \frac{2(e^{x_0} - 1)}{\beta},$$

thus from (32) we obtain

$$(33) \quad g(\beta, \gamma_\beta) = -1 + \alpha K_c\left(\frac{\beta}{\alpha}\right)$$

where for any $x \in]0, 1[$, $K_c(x) = x \ln c + \left(\frac{2}{c} - x\right) \ln\left(1 - \frac{cx}{2}\right) - (1-x) \ln(1-x)$. K_c is strictly concave on $]0, 1[$ and reaches its maximum at $x = \frac{2(c-1)}{c}$.

From (33) with $\frac{\hat{\beta}}{\alpha} = \frac{2(c-1)}{c}$ we get $\max_{\beta > 0} g(\beta, \gamma_\beta) = -1 + \alpha K_c\left(\frac{\hat{\beta}}{\alpha}\right) = -1 + \alpha H(c)$. Then, with (32) we obtain $\gamma_{\hat{\beta}} = \frac{2\alpha}{c} \ln\left(\frac{2\alpha}{2\alpha - \hat{\beta}c}\right) = \frac{-2\alpha \ln(2-c)}{c} := \hat{\gamma}$.

At last, observe that $\frac{\partial g}{\partial \beta}(\beta, \gamma) = \ln\left(\frac{2(e^{x_0} - 1)(\alpha - \beta)}{\beta}\right)$, so $\hat{\beta}$ and $\hat{\gamma}$ give the coordinates of the unique stationarity point of g , that is the unique solution of $\frac{\partial g}{\partial \beta}(\beta, \gamma) = \frac{\partial g}{\partial \gamma}(\beta, \gamma) = 0$.

■

7. Conclusion. We have performed an extensive study of a natural and expressive quantified problem, (1,2)-QSAT. We have proved the existence of a sharp phase transition from satisfiability to unsatisfiability for

(1,2)-QCNF-formulas and we have given the exact location of the threshold. The obtained results have several interesting features. The parameter m , which is the number of universal variables, controls the worst-case computational complexity of the problem (which is ranging from linear time solvable to coNP-complete), as well as the typical behavior of random instances. When m is small, there is a sharp threshold at $c = 2$. On the other side, when m is large enough, actually when $m \gg \ln n$, there is a sharp threshold at $c = 1$: the analysis is similar, and in fact easier, to what we have done for pure snakes in Section 5, in considering snakes with strictly distinct universal variables, as shown in [6]. This fact should be compared to the fact that the threshold location $c^*(\alpha)$ for $m = \lfloor \alpha \ln n \rfloor$ goes to 1 when α goes to infinity. More importantly, an original regime is observed when $m = \lfloor \alpha \ln n \rfloor$. Using counting arguments on pure bicycles, which are the seed of unsatisfiability, and on pure snakes, which are special minimally false formulas, we got respectively a lower and an upper bound for the threshold. It turns out that these two bounds coincide, thus giving the exact location of the threshold as a function of α .

A challenging question would be to determine the scaling window around $c^*(\alpha)$ and get precise information on the typical contradictory cycles that occur in random formulas inside this window.

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